ECE 535
Notes for Lecture # 2

Class Outline:

• Classical Conductivity – Part 2
• Sommerfeld Model - Part 1

Classical Conductivity - 10

We will continue our examination of the Drude Model for metals and refresh our memories about quantum mechanics needed for the Sommerfeld model.

I want to cover:

1. Drude Model for Metals
   • Time-Dependent Dynamics
   • Response Functions

2. Sommerfeld Model
   • Problems with Drude
   • Schrödinger equation
Let's look at a few cases...

**Case I: No Electric Field**

\[ \frac{d\bar{\mathbf{p}}(t)}{dt} = -\frac{\bar{\mathbf{p}}(t)}{\tau} \]

This gives the steady-state solution: \( \bar{\mathbf{p}}(t) = 0 \)

**Case II: Constant and Uniform Electric Field**

\[ \bar{\mathbf{p}}(t) = -e \tau \bar{E} \]

Electron "drift" velocity is defined as:

\[ \bar{\mathbf{v}} = \frac{\bar{\mathbf{p}}(t)}{m} = -\frac{e \tau}{m} \bar{E} = -\mu \bar{E} \]

\[ \mu = e \tau / m = \text{electron mobility} \]

(Units: cm²/V-sec)

**Electron current density** \( \bar{J} \) (Units: Amps/cm²) is:

\[ \bar{J} = n(-e)\bar{\mathbf{v}} = ne \mu \bar{E} = \sigma \bar{E} \]

Where: \( n \) = electron density (Units: #/cm³)

\[ \sigma = \text{electron conductivity} \]

(Units: Siemens/cm) = \( ne\mu = \frac{ne^2\tau}{m} \)

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**Case III: Time Dependent Sinusoidal Electric Field**

But now there is no steady state solution.

- So we assume that the \( \bar{E} \)-field, average momentum, currents are all correspondingly sinusoidal
- We represent them with phasors.

\[ \bar{E}(t) = \text{Re} \left[ \bar{E}(\omega) e^{-i\omega t} \right] \]

\[ \bar{p}(t) = \text{Re} \left[ \bar{p}(\omega) e^{-i\omega t} \right] \]

\[ \bar{J}(t) = \text{Re} \left[ \bar{J}(\omega) e^{-i\omega t} \right] \]

\[ \frac{d\bar{p}(t)}{dt} = -e \bar{E}(t) - \frac{\bar{p}(t)}{\tau} \]

\[ \Rightarrow \bar{p}(\omega) = -\frac{e}{1 - i\omega \tau} \bar{E}(\omega) \]

\[ \bar{v}(\omega) = \frac{\bar{p}(\omega)}{m} = -\frac{e}{1 - i\omega \tau} \bar{E}(\omega) \]

We may now obtain the current density:

\[ \bar{J}(\omega) = n(-e)\bar{v}(\omega) = \sigma(\omega) \bar{E}(\omega) \]

Where

\[ \sigma(\omega) = \frac{ne^2\tau}{m} \left( \frac{1}{1 - i\omega \tau} \right) \]

\[ \sigma(\omega = 0) = \frac{\sigma(\omega = 0)}{1 - i\omega \tau} \]
Let's examine the relationship between current density and the electric field...

\[ \mathbf{J}(\omega) = \sigma(\omega) \mathbf{E}(\omega) \]

This is a relationship between an applied stimulus (the electric field) and the material response (the current density). But we have seen this many times before in different contexts...

\[ \mathbf{P}(\omega) = \varepsilon_0 \chi_e(\omega) \mathbf{E}(\omega) \]

\[ \mathbf{M}(\omega) = \chi_m(\omega) \mathbf{H}(\omega) \]

Let's keep these *linear response functions* in mind as we progress.

Let's now examine another case...

**Case IV: Time Dependent Non-Sinusoidal Electric Field**

For some general time-dependent e-field excitation, we can always simplify our lives by using Fourier transforms.

\[ \mathbf{E}(t) = \int_{-\infty}^{\infty} \mathbf{E}(\omega) e^{i\omega t} \]

\[ \mathbf{E}(\omega) = \int_{-\infty}^{\infty} \mathbf{E}(t) e^{-i\omega t} \]

Then employ the result we obtained for the frequency domain response:

\[ \mathbf{J}(\omega) = \sigma(\omega) \mathbf{E}(\omega) \]

And move back to the time domain:

\[ \mathbf{J}(t) = \int_{-\infty}^{\infty} \mathbf{J}(\omega) e^{i\omega t} \]

\[ = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sigma(\omega) \mathbf{E}(\omega) e^{i\omega t} \]

Substitute (*) into the above result:

\[ \mathbf{J}(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sigma(\omega) \mathbf{E}(\omega) e^{i\omega t} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sigma(\omega) \left[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sigma(\omega) e^{-i\omega(t-t')} \right] \mathbf{E}(t') \]

\[ \Rightarrow \mathbf{J}(t) = \int_{-\infty}^{\infty} dt' \sigma(t-t') \mathbf{E}(t') \]
Let’s examine the last result a bit more...

\[
J(t) = \int_{-\infty}^{\infty} dt' \sigma(t-t') \bar{E}(t')
\]

where

\[
\sigma(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sigma(\omega) e^{-i\omega(t-t')}
\]

- This brings us to an interesting conclusion. The current at some time, \(t\), is a convolution of the conductivity response function and the applied time-dependent electric field.

Consider the Drude model again:

\[
\sigma(\omega) = \frac{\sigma(\omega = 0)}{1 - i\omega \tau}
\]

\[
\sigma(t-t') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sigma(\omega) e^{-i\omega(t-t')} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sigma(\omega = 0) e^{-i\omega(t-t')}
\]

\[
\Rightarrow \sigma(t-t') = \frac{\sigma(\omega = 0)}{\tau} e^{\frac{t-t'}{\tau}} \theta(t-t')
\]

There are two important conditions that linear response functions must satisfy in both the time and the frequency domain:

1) Real inputs must yield real outputs:

Since we had:

\[
J(t) = \int_{-\infty}^{\infty} dt' \left[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sigma(\omega) e^{-i\omega(t-t')} \right] \bar{E}(t')
\]

This condition can only hold if:

\[
\sigma(-\omega) = \sigma^*(\omega)
\]

2) Output must be causal (i.e. output at any time cannot depend on future input):

Since we had:

\[
J(t) = \int_{-\infty}^{\infty} dt' \sigma(t-t') \bar{E}(t')
\]

This condition can only hold if:

\[
\sigma(t-t') = 0 \text{ for } t < t'
\]

Both of these conditions are satisfied by the Drude model, so that’s a good thing.
Can the Drude model tell us anything about waves incident on a metal?

For instance, we already know that EM waves incident from air onto a metal will have a reflected component...

$$\varepsilon_0 \quad \mu_0$$

Basic undergrad EM tells us the reflection coefficient:

$$\Gamma = \frac{E_r}{E_i} = \frac{\varepsilon_0 - \varepsilon(\omega)}{\sqrt{\varepsilon_0 + \varepsilon(\omega)}}$$

What is the relative permittivity for a metal as a function of frequency?

Start from Maxwell's equations:

Ampere's law:

$$\nabla \times \mathbf{H}(\mathbf{r},t) = \mathbf{J}(\mathbf{r},t) + \varepsilon_0 \frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}$$

Write it in phasor form:

Phasor form:

$$\nabla \times \mathbf{H}(\mathbf{r}) = \mathbf{J}(\mathbf{r}) - i \omega \varepsilon_0 \mathbf{E}(\mathbf{r})$$

$$= \sigma(\omega) \mathbf{E}(\mathbf{r}) - i \omega \varepsilon_0 \mathbf{E}(\mathbf{r})$$

$$= -i \omega \varepsilon_0 \sigma(\omega) \mathbf{E}(\mathbf{r})$$

The metal reflection coefficient becomes:

$$\Gamma = \frac{E_r}{E_i} = \frac{\sqrt{\varepsilon_0 - \varepsilon_{\text{eff}}(\omega)}}{\sqrt{\varepsilon_0 + \varepsilon_{\text{eff}}(\omega)}}$$

Now use the Drude expression:

$$\sigma(\omega) = \frac{\sigma(\omega = 0)}{1 - i \omega \tau}$$

Using this, we can explain the frequency dependence of the reflection coefficient of metals from RF to optical frequencies.
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For metals we have the following two results:

\[ \varepsilon_{\text{eff}}(\omega) = \varepsilon_0 \left( 1 + \frac{i \sigma(\omega)}{\omega \varepsilon_0} \right) \quad \text{and} \quad \sigma(\omega) = \frac{n e^2 \tau}{m} \]

Let's examine the frequency response for small frequencies: \((\omega \tau << 1)\)

\[ \sigma(\omega) \approx \sigma(\omega = 0) = \frac{n e^2 \tau}{m} \quad \Rightarrow \quad \varepsilon_{\text{eff}}(\omega) \approx \varepsilon_0 \left( 1 + \frac{i \sigma(\omega = 0)}{\omega \varepsilon_0} \right) \]

For large frequencies, we see something interesting: \((\omega \tau >> 1)\)

\[ \sigma(\omega) \approx \sigma(\omega = 0) = \frac{n e^2}{m \omega} \quad \Rightarrow \quad \varepsilon_{\text{eff}}(\omega) \approx \varepsilon_0 \left( 1 - \frac{\omega_p^2}{\omega^2} \right) \]

\[ \omega_p = \sqrt{\frac{n e^2}{\varepsilon_0 m}} \]

The electrons behave like a plasma free of collisions.

*Note that for \(\omega_p > \omega >> \frac{1}{\tau}\) the dielectric constant is real and negative.*

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**Classical Conductivity – 19**

Consider a metal with electron density, \(n\), and assume all the electrons in a certain region are displaced by, \(u\).

\[ \text{The electric field generated: } E = \frac{n e u}{\varepsilon_0} \]

\[ \text{The force on the electrons: } F = -eE = -\frac{n e^2 u}{\varepsilon_0} \]

Displacement obeys Newton's second law:

\[ m \frac{d^2 u(t)}{dt^2} = F = -eE = -\frac{n e^2 u(t)}{\varepsilon_0} \quad \Rightarrow \quad \frac{d^2 u(t)}{dt^2} = -\omega_p^2 u(t) \]

Solution is: \(u(t) = A \cos(\omega_p t) + B \sin(\omega_p t)\)

*Charge density oscillations.*
One obvious problem is that we have, thus far, ignored scattering. From the Drude model, we know that:
\[
\frac{d\vec{p}(t)}{dt} = -e \vec{E}(t) - \frac{\vec{p}(t)}{\tau} \Rightarrow m \frac{d^2u(t)}{dt^2} = -e \vec{E}(t) - \frac{m}{\tau} \frac{du(t)}{dt}
\]

We also know the electric field:
\[
E(t) = \frac{ne\ u(t)}{\varepsilon_0}
\]

Combine the two equations to get a new differential equation:
\[
\frac{d^2u(t)}{dt^2} = -\omega_p^2 \ u(t) - \frac{1}{\tau} \frac{du(t)}{dt}
\]
\[
\omega_p = \sqrt{\frac{ne^2}{\varepsilon_0 m}}
\]

\[
\frac{d^2u(t)}{dt^2} + \frac{1}{\tau} \frac{du(t)}{dt} + \omega_p^2 \ u(t) = 0
\]

\[\text{second order system with damping} \]

Let’s examine two cases of plasma oscillations in the presence of scattering:

Case I: Underdamped oscillations \( \omega_p > \frac{1}{2\tau} \)
\[
u(t) = e^{-\gamma t} [A \cos(\Omega_p t) + B \sin(\Omega_p t)]
\]

Where:
\[
\gamma = \frac{1}{2\tau} \quad \Omega_p = \sqrt{\omega_p^2 - \gamma^2}
\]

Case II: Overdamped oscillations \( \omega_p < \frac{1}{2\tau} \)
\[
u(t) = A e^{-\gamma_1 t} + B e^{-\gamma_2 t}
\]

Where:
\[
\gamma_1 = \frac{1}{2\tau} + \sqrt{\frac{1}{4\tau^2} - \omega_p^2} \quad \gamma_2 = \frac{1}{2\tau} - \sqrt{\frac{1}{4\tau^2} - \omega_p^2}
\]
Sommerfeld Model - 1

But...the Drude theory has some problems...

\[ \frac{dp(t)}{dt} = m \frac{dv(t)}{dt} = F - \frac{mv(t)}{\tau} \]

\[ \frac{dp(t)}{dt} = -e \left[ E + v(t)xB \right] - \frac{mv(t)}{\tau} \]

- Does not say anything about the electron energy distribution in metals
- Are all electrons moving around with about the same energy?

- Does not take into account Pauli’s exclusion principle

To account for these shortcomings Sommerfeld in 1927 developed a model for electrons in metals that took into consideration the Fermi-Dirac statistics of electrons

Note added:

Six of Sommerfeld’s students - Werner Heisenberg, Wolfgang Pauli, Peter Debye, Hans Bethe, Linus Pauling, and Isidor I. Rabi - went on to win Nobel prize in Physics.

He was nominated 84 times for the Nobel Prize but never won...most ever.