Problem # 1

Consider the metallic system shown above. The metal has an electron density, \( n \), and an electron scattering time, \( \tau \). Now apply a uniform magnetic field in the \( z \)-direction of magnitude \( \vec{B} = B_0 \hat{z} \) in addition to the normal longitudinal (\( x \)-directed) applied electric field, \( E_x \). In the presence of these crossed fields, we know that the force imparted to the electrons is given by the Lorentz force, \( \vec{F} = -q (\vec{E} + \vec{v} \times \vec{B}) \). Keeping this in mind, please answer the following questions:

a) Suppose that, in this material, the scattering rate is zero and that we have turned off the electric field. Please use the Drude model to find the average electron velocity in both the \( x \) and \( y \) directions assuming initial conditions such that \( v_x(t = 0) = A \) and \( v_y(t = 0) = 0 \). **HINT:** You should obtain two coupled linear differential equations for the two components of velocity.

b) The components of the average electron displacement, \( u_x(t) \) and \( u_y(t) \), are related to the average electron velocities by \( \frac{du_x(t)}{dt} = v_x(t) \) and \( \frac{du_y(t)}{dt} = v_y(t) \). Now that you have obtained the average velocities, use the initial conditions \( u_x(t = 0) = u_y(t = 0) = 0 \) to calculate the average electron displacement. What does your solution tell you about the electron motion? **HINT:** the motion should be oscillatory. What is the angular frequency? What is the radius of this motion? Is the motion clockwise or counter clockwise if one were to look down at the top of the metal?

c) Now turn back on the electric field, \( E_x \), and repeat parts (a) and (b). The presence of a driving term will require you to add back in a particular solution to the previously found coupled linear differential equations. In this situation, you should find that the motion of the electron is helical, or circular motion with an average uniform velocity in a certain direction. What direction is this in our case?

d) Assume now that both \( E_x \) and \( E_y \) are non-zero. Even though we have not explicitly applied an electric field in the \( y \)-direction, you will find that a \( y \)-directed electric field must exist in our system. Let us now turn the scattering back on in our system. This corresponds to the most realistic of our situations. To find the full time-dependent solution to this problem is too tedious...
and not instructive so let us limit ourselves to the time-independent solution, which comes about due to the presence of the scattering term. To find this solution, recall that the Drude model states that \( m \frac{\partial \vec{v}}{\partial t} = -q (\vec{E} + \vec{v} \times \vec{B}) - m \frac{\vec{v}}{\tau} \) but, as we are interested in steady-state, \( \frac{\partial \vec{v}}{\partial t} = 0 \). What are \( v_x \) and \( v_y \) in steady-state as a function of the components of the electric and magnetic fields?

e) What are the directional components of the current density, \( J_x \) and \( J_y \)? Interestingly, you should find a non-zero component to \( J_y \).

f) Our system is finite in the y-direction and there are no leads present on the sides, so there cannot be steady state current flowing in the y-direction inside the sample. Therefore, the only way to have \( J_y = 0 \) is to have a non-zero component of \( E_y \). What are the magnitude and direction of the field \( E_y \)?

g) A non-zero \( E_y \) can be, and routinely is, measured experimentally by using voltage probes on both sides of the y-direction that are spaced by a distance, \( d \), apart, as we have shown in the figure. The ratio \( \frac{E_y}{J_x} \) is called the Hall resistivity, \( \rho_{Hall} \). Find the magnitude and sign of \( \rho_{Hall} \).

Problem # 2

a) In gold, the electron scattering time, \( \tau \), is approximately 30 fs, or \( 30 \times 10^{-15} \) seconds. You can find the DC conductivity of gold on the website: www.webelements.com. Find the carrier density, \( n \), of electrons in gold. Find the plasma frequency, \( \omega_p \), of gold and then, using the figure below, determine in which part of the spectrum it falls.

![Spectrum of Electromagnetic Radiation](image)

b) In a typical doped semiconductor with dielectric constant \( \varepsilon_0 \), the electron density is \( n = 1017 \) cm\(^{-3}\). Find the plasma frequency, \( \omega_p \), of this semiconductor and then, using the figure below, determine in which part of the spectrum it falls.
Problem # 3
Consider a quantum system with the Hamiltonian operator, $H_0$. The Hamiltonian has two eigenstates, $|\phi_1\rangle$ and $|\phi_2\rangle$ with the same eigenenergies, $E$, so that $H_0|\phi_1\rangle = E|\phi_1\rangle$ and $H_0|\phi_2\rangle = E|\phi_2\rangle$.

a) Suppose that we add a perturbing term, $H'$, is added to the Hamiltonian so that we have a new Hamiltonian of $H = H_0 + H'$. The matrix elements of the perturbing term, $H'$, are: $\langle\phi_2|H'|\phi_1\rangle = \langle\phi_1|H'|\phi_2\rangle = t$. Try to find a trial solution for the eigenstates of $H$, which is a superposition of the eigenstates of $H_0$ as $|\psi\rangle = c_1|\phi_1\rangle + c_2|\phi_2\rangle$ so that $H|\psi\rangle = E|\psi\rangle$. Plug in the trial solution and find all possible eigenvalues and the corresponding eigenstates of the Hamiltonian, $H$.

b) In the basis of the eigenstates, the Hamiltonian, $H_0$, is diagonal and can be represented by the matrix:

$$H_0 = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}$$

and the two eigenstates can be represented by two corresponding column vectors, $|\phi_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|\phi_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. As before, we add a perturbing term to the Hamiltonian so that the new Hamiltonian has the form:

$$H = H_0 + H' = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} + \begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix} = \begin{bmatrix} E + t & t \\ t & E \end{bmatrix}$$

Please find the eigenstates and eigenenergies of the new Hamiltonian in terms of $|\phi_1\rangle$ and $|\phi_2\rangle$.

Problem # 4

a) Consider a periodic function that consists of a train of delta functions of equal weights separated in time by $T$, $f(t) = \sum_{n=0}^{\infty} \delta(t - nT)$. Please find the Fourier transform $f(\omega)$ and show that it also consists of a train of delta functions of equal weight in the frequency domain.

b) Consider the box below and find the Fourier transform $g(\omega)$.

\[ g(t) = \begin{cases} 1 & \text{if } 0 < t < T \\ 0 & \text{otherwise} \end{cases} \]
c) Consider the periodic function below:

\[ h(t) = \]

\[ T \quad 1 \quad T \quad \]  

Find the Fourier transform, \( h(\omega) \).

d) Use your results from (c) to show that \( h(t) \) can be written as a Fourier series: 
\[ h(t) = \sum_{-\infty}^{\infty} h_n \cdot \frac{1}{T} e^{-i \omega n t} \] and find the coefficients \( h_n \).

e) Consider the function \( h(x,y,z) \) in three dimensions. The function \( h(x,y,z) \) is equal to 1 inside a cube centered at the origin and of dimensions shown in the figure below, and is equal to zero outside the cube. Find the Fourier transform \( h(k_x, k_y, k_z) \).