ECE 340
Lecture 7: Fermi Level and Equilibrium Carrier Distributions

Class Outline:

• Density of States
• Fermi Level
• Equilibrium Carrier Distributions

Things you should know when you leave...

Key Questions

• What is the density of states?
• What does the Fermi function tell us?
• What is the Fermi level?
• How do I find the equilibrium electron and hole concentrations?
• When can I use the simplified equations for the concentrations?

Density of States

We are counting states...

The total number of states in a small region of volume \( \Delta k \) can be written as

\[
N_{\text{total}}(k) = \frac{2}{\hbar^3} \Delta k.
\]

Consider free electrons, the volume of the sphere between the two spheres:

\[
\Delta k = 4 \pi k^2 \Delta k.
\]

The separation \( \Delta k \) corresponds to an energy difference (for free electrons):

\[
\Delta E = \frac{2m}{\hbar^2} k^2 \Delta k.
\]

Above:

• Constant energy surfaces in 3D k-space.
• \( \Delta k \) is the volume of the sphere contained between the two spheres of radius \( k \) and \( k + \Delta k \).

Energy

\[ \begin{align*}
\text{Available Conduction Band States} & : E_c \\
\text{Available Valence Band States} & : E_v
\end{align*} \]

Density of States

But it is in k-space...

The number of states in the shell is given in terms of the density of states in energy by:

\[
N(E) = \frac{2}{\hbar^3} 4 \pi k^2 \Delta k.
\]

Plug in what we know about how the free electron model changes in k as we change E then integrate to get the final result:

\[
N(E) = \frac{m^*}{\pi \hbar^2} \sqrt{2m E - E_f}.
\]
Density of States

1. Density of states (DOS) is like seating at a stadium with the number of seats a given distance from the field being available in the range of energy $E - E_c < 0$.
2. In general, the number of states available in the conduction and valence bands will not be equal.
3. Considering closely spaced energies $E$ and $E + dE$ in the conduction band, we can state:
   - $f(E)$ is the number of valence band states lying in the energy range $E - E_c < 0$.
   - $f(E + dE)$ is the number of conduction band states lying in the energy range $E + dE > 0$.

Fermi Level

What about the Fermi Level?

- What observations can we make about temperature dependence?
  - At $E = E_f$, $f(E) = \frac{1}{2}$.
  - At $E = E_f + 3k_B T$, $f(E)$ decays rapidly to zero and most states will be empty.
  - At $E = E_f - 3k_B T$, $1 - f(E)$ decays rapidly to zero. Therefore, most states will be filled.

Fermi Level

Fine, but how many states are ACTUALLY filled?

- The Fermi function tells us the probability of how many of the existing states at any given energy will be filled with an electron.
- It was derived based on three assumptions:
  - Each allowed state has a maximum of one electron (Pauli Exclusion Principle).
  - All electrons are indistinguishable.
- When is it valid?
  - Under equilibrium conditions.
  - In ALL materials (semiconductor, metal, insulator, etc.).

Mathematically speaking the end result is a probability distribution function:

$$f(E) = \begin{cases} \frac{1}{1 + e^{(E - E_f)/k_B T}} & \text{if } E - E_f > 0 \\ \frac{1}{1 + e^{-(E - E_f)/k_B T}} & \text{if } E - E_f < 0 \end{cases}$$

Fermi Energy

Let's look at a quick example...

- Let's say that the probability that a state is filled at the conduction band edge is equal to the probability that a state is empty in the valence band. Where is the Fermi level?

$$f(E_c) = 1 - f(E_v) = \frac{1}{1 + e^{-E_c/k_B T}}$$
Fermi Level

**Visualizing the Fermi level in intrinsic material.**

- In intrinsic material:
  - Concentration of holes in valence band is equal to the number of electrons in the conduction band.
  - Electron probability tail, \( f(E) \), is symmetric with hole probability tail, \( 1 - f(E) \).
  - No states in \( E_g \) despite non-zero occupation probability.
  - Fermi level (\( E_f \)) lies in the middle of the energy gap.

**Carrier Distributions:**

- For electrons:
  \[ n = \int_{E_f}^{E_c} f(E) n(E) \ dE \]
- For holes:
  \[ p = \int_{E_v}^{E_f} f(E) n(E) \ dE \]

**Equilibrium Carrier Distributions**

Let’s put together our knowledge of the density of states and the Fermi function.

We want to know how many carriers we have in our semiconductor for ANY energy or temperature.

For electrons:
\[ n = \frac{1}{\pi^2} \int_{E_f}^{E_c} \frac{1}{(E-E_f)^2 + (\pi \hbar v)^2} \ dE \]
Focus on the electron relation.

For holes:
\[ p = \frac{1}{\pi^2} \int_{E_v}^{E_f} \frac{1}{(E-E_v)^2 + (\pi \hbar v)^2} \ dE \]
Plug in the relationships for the density of states and the Fermi function.

Make some substitutions and bring the constants out:
\[ n = \frac{m^*}{\pi^2 \hbar^2} \sqrt{\frac{2m^*}{\epsilon_0 T}} \sqrt{\frac{E-E_f}{\epsilon_0 T}} \frac{E-E_f}{\epsilon_0 T} \ E_m = \frac{m^*}{\pi^2 \hbar^2} \sqrt{\frac{2m^*}{\epsilon_0 T}} \sqrt{\frac{E-E_f}{\epsilon_0 T}} \]
That integral is awful! Isn't there an easier way?

\[ N_c \quad \text{Fermi-Dirac integral of order} \frac{3}{2} \]

\[ N_v \quad \text{Fermi-Dirac integral of order} \frac{3}{2} \]

\[ \eta = 2 \left( \frac{m_c k_B T}{\hbar} \right)^{3/2} \]

\[ \eta = 2 \left( \frac{m_v k_B T}{\hbar} \right)^{3/2} \]

\[ \eta = 2 \left( \frac{m_l k_B T}{\hbar} \right)^{3/2} \]

\[ \text{Effective conduction band density of states.} \]

\[ \text{Effective valence band density of states.} \]

\[ \text{Non-degenerate} \]

\[ \text{Degenerate} \]

\[ \eta = 2 \left( \frac{m_e k_B T}{\hbar} \right)^{3/2} \]

\[ \eta = 2 \left( \frac{m_h k_B T}{\hbar} \right)^{3/2} \]

\[ \eta = 2 \left( \frac{m_t k_B T}{\hbar} \right)^{3/2} \]

\[ \text{Non-degenerate Semiconductor} \]

\[ \text{Degenerate Semiconductor} \]

\[ n = n_e \left( \frac{k_f}{k_B T} \right)^{3/2} \]

\[ p = p_e \left( \frac{k_f}{k_B T} \right)^{3/2} \]

\[ N_c = N_e \left( \frac{k_f}{k_B T} \right)^{3/2} \]

\[ N_v = N_v \left( \frac{k_f}{k_B T} \right)^{3/2} \]

\[ N_l = N_l \left( \frac{k_f}{k_B T} \right)^{3/2} \]

By combining equations, we arrive at two very important and useful results:

\[ n = n_e \left( \frac{k_f}{k_B T} \right)^{3/2} \]

\[ p = p_e \left( \frac{k_f}{k_B T} \right)^{3/2} \]

\[ N_c = N_e \left( \frac{k_f}{k_B T} \right)^{3/2} \]

\[ N_v = N_v \left( \frac{k_f}{k_B T} \right)^{3/2} \]

\[ N_l = N_l \left( \frac{k_f}{k_B T} \right)^{3/2} \]

A note on the effective mass:

- In silicon, most of the electrons reside in the X valley as it is lowest in energy.
- The X direction has six equivalent valleys with different masses in each of the possible directions.
- We use the effective mass which is the geometric mean of the masses (density of states effective mass).
- GaAs conduction band does not have this complication.